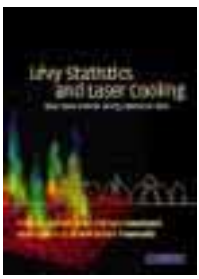

Lévy Statistics and Laser Cooling

How Rare Events Bring Atoms to Rest

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Chapter

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2

Subrecoil laser cooling and anomalous random walks

In this chapter, we first recall (in Section 2.1) a few properties of the most usual laser cooling schemes, which involve a friction force. In such standard situations, the motion of the atom in momentum space is a Brownian motion which reaches a steady-state, and the recoil momentum of an atom absorbing or emitting a single photon appears as a natural limit for laser cooling. We then describe in Section 2.2 some completely different laser cooling schemes, based on inhomogeneous random walks in momentum space. These schemes, which are investigated in the present study, allow the 'recoil limit' to be overcome. They are associated with non-ergodic statistical processes which never reach a steady-state. Section 2.3 is devoted to a brief survey of various quantum descriptions of subrecoil laser cooling, which become necessary when the 'recoil limit' is reached or overcome. The most fruitful one, in the context of this work, is the 'quantum jump description' which will allow us in Section 2.4 to replace the microscopic quantum description of subrecoil cooling by a statistical study of a related classical random walk in momentum space. It is this simpler approach that will be used in the subsequent chapters to derive some quantitative analytical predictions, in cases where the quantum microscopic approach is unable to yield precise results, in particular in the limit of very long interaction times, and/or for a momentum space of dimension D higher than 1. This approach also has the advantage of showing, from the beginning, that anomalous random walks and Lévy statistics are deeply involved in subrecoil laser cooling.

2.1 Standard laser cooling: friction forces and the recoil limit

2.1.1 Friction forces and cooling

Laser cooling consists of using resonant exchanges of linear momentum between atoms and photons for reducing the momentum spread δp of an ensemble of atoms. Achieving the smallest possible value of δp has led to spectacular developments

in various research fields such as metrology, high resolution spectroscopy, atom optics, atomic interferometry, Bose–Einstein condensation of atomic gases, ...

In the most usual laser cooling schemes, an ensemble of atoms interacts with laser beams with suitable polarizations, intensities and frequencies, so that the atomic momenta are damped. In the so-called 'Doppler cooling' scheme for instance [HaS75, WiD75], this damping is due to an imbalance between opposite radiation pressure forces, induced by the Doppler effect and resulting in a net force opposed to the atomic motion. In the more efficient 'Sisyphus cooling' scheme [DaC89, UWR89, CoP90], atoms run up potential hills (where they decelerate) more frequently than they run down, with the net result of a decrease of atomic kinetic energy.

In all these situations, the cooling effect can be expressed by a friction force which, around $\mathbf{p} = \mathbf{0}$, is proportional to the atomic momentum $\mathbf{p}$ with a negative coefficient [Coh90]. When the friction coefficient is large, atoms seem to move in a very viscous medium, and this kind of situation is called 'optical molasses' [CHB85, LPR89].

Momentum damping by a friction force is a dissipative process, necessarily associated with some fluctuations. In the laser cooling schemes considered above, the fluctuations are due to spontaneous emission of fluorescence photons which can be emitted at random times and in random directions, resulting in a random fluctuating component of the momentum exchanged between the atom and the radiation field. Laser cooling mechanisms associated with a friction force therefore give rise to a random walk of the atomic momentum $\mathbf{p}$. As in usual Brownian motion, such a random walk can be characterized by a drift of the atomic momentum towards $\mathbf{p} = \mathbf{0}$ due to the friction force (damping of the mean momentum), and by a momentum diffusion, due to the randomness introduced here by spontaneous emission. Competition between friction and diffusion eventually leads to a steady-state, where the momentum distribution can be characterized by a stationary probability distribution of half-width δp. However, even in the steady-state, fluorescence cycles never cease, and the random walk in momentum space never stops.

Even in the steady-state of the cooling process, laser cooled gases are generally not in thermal equilibrium and, as such, cannot be characterized by a well defined thermodynamic temperature. It is, however, convenient to express the half-width δp of the distribution using an 'effective temperature' T defined by

$$\frac{1}{2}\, k_{\mathrm{B}} T = \frac{\delta p^2}{2M}, \tag{2.1}$$

where k_{B} is the Boltzmann constant, M is the atomic mass and δp is the half-width at $\mathrm{e}^{-1/2}$ of the maximum of the one-dimensional momentum distribution. When the momentum distribution is Maxwell–Boltzmann, the definition of eq. (2.1)

coincides with the usual definition of statistical mechanics. Throughout this book, we will use notations T and δp as defined by eq. (2.1).

2.1.2 The recoil limit

What is the minimum temperature that can be reached by standard laser cooling mechanisms? Since fluorescence cycles never cease and involve spontaneously emitted photons which communicate to the atom a random recoil, it seems difficult to control the momentum spread δp to better than the 'recoil limit' $\hbar k$, the momentum of a single photon. In terms of temperature, the recoil limit reads:

$$T \geq T_{\mathrm{R}}, \tag{2.2}$$

where the 'recoil temperature' T_{R} is defined by

$$T_{\mathrm{R}} = \frac{\hbar^2 k^2}{M k_{\mathrm{B}}}. \tag{2.3}$$

The quantity $k_{\mathrm{B}} T_{\mathrm{R}}/2$ is the kinetic energy given to an atom at rest by the absorption or the emission of a single photon. For most atoms, the recoil temperature is of the order of 1 microkelvin (μK). Optimized Sisyphus cooling leads to temperatures close to the recoil limit [CaM95].

2.2 Laser cooling based on inhomogeneous random walks in momentum space

2.2.1 Physical mechanism

It is also possible to achieve laser cooling, i.e. to accumulate atoms around the origin $\mathbf{p} = \mathbf{0}$ of the momentum space, without any friction force. Rather than pushing the atoms towards $\mathbf{p} = \mathbf{0}$ (drift of the momentum random walk associated with the friction), one resorts to an inhomogeneous diffusion coefficient, vanishing around $\mathbf{p} = \mathbf{0}$, so that the random walk slows down in this region, where the atoms pile up. Although random walks seem at first sight to be less efficient at reducing momenta than the deterministic trend provided by friction, it is these random walks that will enable one to circumvent the recoil limit.

Such a situation is achieved when the fluorescence rate R, at which photons are spontaneously re-emitted, depends on $\mathbf{p}$ and exactly vanishes for atoms with zero momentum $\mathbf{p} = \mathbf{0}$ (Fig. 1.1a).

The consequence of the vanishing of the fluorescence rate $R(\mathbf{p})$ around $\mathbf{p} = \mathbf{0}$ is that ultracold atoms ($p \simeq 0$) no longer undergo the random recoils which would be due to spontaneous emissions. They are in some sense protected from the 'bad' effects of light. On the other hand, atoms with $p \neq 0$ can absorb

light. The random recoil due to the re-emitted photons modifies in a random way their momentum, which can move closer to zero, or farther from zero, after each fluorescence cycle. The **p**-dependent fluorescence rate of Fig. 1.1a is thus at the origin of an inhomogeneous random walk in **p**-space, with a **p**-dependent jump rate $R(\mathbf{p})$, which eventually transfers atoms from the $p \neq 0$ absorbing states into the $p \simeq 0$ non-absorbing 'dark' states where they remain trapped and pile up (Fig. 1.1b).

A simple analogy can help understanding such a physical mechanism. Consider sand grains in a Kundt tube where a resonant acoustic standing wave is excited. The sand grains are vibrating and moving along the axis of the standing wave in an erratic way, except at the nodes of the standing wave, where there is no sound vibration to excite them. After a certain time, the sand grains accumulate at the nodes of the standing wave. In both cases, cooling without friction and Kundt's tube, we have an inhomogenous random walk, i.e. a random walk with a jump rate varying with the location of the particle and vanishing at certain places. For cooling, however, the random walk is in momentum space and the jump rate is momentum-dependent, whereas in a Kundt's tube, the sand grains move in real space, and the jump rate is position-dependent.

The fact that the cooling processes considered here do not rely on a friction force does not mean of course that they could not benefit from the presence of such a friction force. Without friction, the cooling relies on the efficiency with which a pure random walk can bring an atom near $\mathbf{p} = \mathbf{0}$. Such an efficiency decreases dramatically with the number D of dimensions. In one dimension, every particle returns often to the origin of momentum space, although it may take a very long time; in two dimensions, this return takes a much longer time and in three dimensions returning to the origin becomes even more unlikely. It is thus very useful to supplement the momentum random walk with a friction force, producing a drift that tends to push the atoms towards the origin of momentum space [MaA91]. This clearly improves the accumulation process. All subrecoil cooling schemes that have been implemented offer the possibility of an efficient friction force.

2.2.2 How to create an inhomogeneous random walk

Up to now, two methods of laser cooling based on the inhomogeneous random walk presented above have been proposed and demonstrated: Velocity Selective Coherent Population Trapping (VSCPT) [AAK88] and Raman cooling [KaC92]. In this book, we do not present a detailed description of these two methods[1], which can be found elsewhere [AAK89, Coh90, Ari91, OlM90, MaA91, KaC92, Rei96,

[1] However, Appendix A gives a detailed derivation of the parameters of inhomogeneous random walks corresponding to VSCPT and Raman cooling.

RSC01]; we just indicate how these methods fit into the scheme discussed in this work.

In Coherent Population Trapping, an atom with several ground state sublevels interacts with a set of quasi-resonant laser beams, such that there exists a particular linear superposition of these ground state sublevels where the atom does not interact with the laser light. This cancellation of the coupling is due to a destructive quantum interference between absorption amplitudes, and it leads to the accumulation of the atoms into the uncoupled state where the atoms cease to fluoresce ('dark state') and become trapped [AGM76, ArO76]. A careful analysis taking into account the quantization of atomic motion [AAK89, Coh90, Coh96] shows that, for most laser configurations, the cancellation of the fluorescence rate actually depends on a generalized atomic momentum $\mathbf{p}$ (which is now a quantum number) and that situations exist where $R(\mathbf{p})$ exactly cancels at $\mathbf{p} = \mathbf{0}$. The fluorescence rate thus becomes 'Velocity Selective'. In this VSCPT process, one can show that $R(\mathbf{p})$ varies quadratically with $\mathbf{p}$ around $\mathbf{p} = \mathbf{0}$. On the other hand, the behaviour of $R(\mathbf{p})$ at large $\mathbf{p}$ depends on the laser configuration, but it usually saturates and decreases asymptotically as p^{-2} because of the Doppler shift. Most VSCPT schemes give rise to a friction force for a proper sign of the detuning between the laser frequency and the atomic frequency [SHP93, MDT94, WEO94, HLO00]. Notice, however, that the initial VSCPT scheme [AAK88] involves no such force and appears to be an example of cooling due purely to an inhomogeneous random walk.

In Raman cooling, an atom with two hyperfine ground state sublevels interacts with a sequence of laser pulses leading to population transfers between the ground state sublevels, through stimulated and spontaneous Raman processes. Here also, this process can be made momentum-dependent, and it is possible to choose sequences of pulses such that atoms stop interacting with the lasers at $\mathbf{p} = \mathbf{0}$. The $\mathbf{p}$-dependence of the fluorescence rate $R(\mathbf{p})$ around $\mathbf{p} = \mathbf{0}$, as well as for large values of $\mathbf{p}$, can even be tailored almost at will [KaC92, RBB95]. Moreover, a friction force is readily implemented by an appropriate choice of the directions of the laser beams used for the stimulated Raman transitions [KaC92].

2.2.3 Expected cooling properties

The exact vanishing of the random walk jump rate $R(\mathbf{p})$ at $\mathbf{p} = \mathbf{0}$, which is the very basis of the non-standard cooling mechanisms considered here, has important consequences.

A first consequence of the vanishing of $R(\mathbf{p})$ around $\mathbf{p} = \mathbf{0}$ is the absence of a steady-state, even at arbitrarily long interaction times θ. Indeed, let us consider the

characteristic evolution time $\tau(\mathbf{p})$ defined by

$$\tau(\mathbf{p}) = R(\mathbf{p})^{-1}, \tag{2.4}$$

i.e. the mean time for an atom with momentum $\mathbf{p}$ to undergo a fluorescence cycle, and let us compare it to the 'interaction time' θ, a key quantity in this study, which represents the duration of the interaction between the atoms and the laser beams that cool them. Since $R(p) \underset{p\to 0}{\to} 0$, one has $\tau(p) \underset{p\to 0}{\to} \infty$. Therefore, however long the interaction time θ may be, there exists atomic evolution times $\tau(\mathbf{p})$ longer than θ, namely the ones corresponding to $p < p_\theta$ where p_θ is defined by

$$R(p_\theta)\,\theta = 1. \tag{2.5}$$

We will show in this work that these unbounded evolution times can introduce a fundamental non-ergodicity in the problem, hence the denomination 'non-ergodic cooling' used in this book.

As a second consequence, the single photon recoil $\hbar k$ is no longer a fundamental limit for cooling. In this study, we will show that the radius δp of the volume around $\mathbf{p} = \mathbf{0}$ where the atoms are accumulated depends only on the shape of $R(\mathbf{p})$ around $\mathbf{p} = \mathbf{0}$, and on the interaction time θ between the atoms and the lasers. The radius δp will be shown to be of the order of p_θ, defined according to eq. (2.5) as the smallest momentum for which the probability of a jump during θ is not negligible. Since $R(p)$ increases monotonically with p, this relation leads to values of $\delta p \simeq p_\theta$ which decrease indefinitely with θ, and thus become smaller than the recoil $\hbar k$ for long enough interaction times. This is how the recoil limit has been overcome in laser cooling experiments based on the schemes described here [AAK88, KaC92, LBS94, LKS95, RBB95].

The key role played by the very long sojourn times around $\mathbf{p} = \mathbf{0}$ is the basic ingredient of cooling by inhomogeneous random walks. In fact, the probability distribution of the sojourn times τ between successive jumps, has such long tails for large values of τ, that the variance or even the average of τ may not exist. This is why the inhomogeneous random walks associated with these cooling mechanisms are anomalous, and cannot be treated by standard statistics.

2.3 Quantum description of subrecoil laser cooling

2.3.1 Wave nature of atomic motion

The description of subrecoil cooling given in Section 2.2 is oversimplified. In the subrecoil regime, the external degrees of freedom, i.e. the position x and the momentum p of the atomic centre of mass, must be treated quantum-mechanically (we take one-dimensional notations for simplicity). This is due to the fact that the

condition $\delta p < \hbar k$ is equivalent to $h/\delta p > h/(\hbar k)$, i.e. to $\xi > \lambda_L$ where $\xi = h/\delta p$ is the atomic spatial coherence length and $\lambda_L = h/(\hbar k)$ is the laser wavelength. In other words, below the recoil limit, the spatial coherence length of atomic wave packets becomes larger than the wavelength of the lasers which are used to cool the atoms, so that it becomes impossible to make the semiclassical approximation consisting of a classical treatment of the position of the atomic centre of mass [Coh90]. In this regime, the wave nature of the atomic motion can no longer be ignored, and the description of the atomic state must include not only internal quantum numbers g_m, but also external quantum numbers q labelling the motion of the centre of mass. Moreover, we want to describe the evolution of an ensemble of atoms submitted to dissipation.

In order to take into account both the wave nature of the atomic motion and the dissipation, the quantum description generally uses a density matrix σ, the elements of which $\langle g_m, q | \sigma | g_{m'}, q' \rangle$ are labelled by internal and external quantum numbers. The equations of motion of this infinite set of density matrix elements (including those which involve the Zeeman sublevels of the excited state manifold e) are the so-called Generalized Optical Bloch Equations (GOBE).

2.3.2 *Difficulties of the standard quantum treatment*

Because of their complexity, it is in general impossible to obtain analytical solutions of the GOBE. Furthermore, in the subrecoil laser cooling mechanisms considered here, there is no steady-state solution, and it is thus difficult to make simple predictions concerning the long time regime where these mechanisms are most interesting. There are, however, a few exceptions for simple one-dimensional laser configurations, where analytical predictions for asymptotic behaviour have been obtained [AlK92, AlK93, AlK96, SSY97]. Unfortunately, it seems to be extremely difficult to extend these methods to more general laser configurations and to higher dimensions.

A numerical integration of the GOBE also raises serious practical problems in the limit of long times. Suppose that one chooses momentum eigenvalues p for labelling the state of the centre of mass. To make numerical calculations, one must discretize p over a grid which has to be finer and finer as time grows, because of the appearance of a narrow peak in the momentum distribution, with the width p_θ tending to zero when θ tends to infinity. But in the same limit (and in the absence of friction), the momentum diffusion also spreads the possible values of p over a larger and larger interval which grows with increasing θ. It follows that the number of equations to be numerically solved becomes prohibitively large, especially in dimensions larger than one. It is therefore impossible to make precise calculations for long times, and to obtain information concerning, for example, the proportion

of cooled atoms contained in the narrow peak of the momentum distribution (see, for example, Section 6.F of [AAK89]).

2.3.3 Quantum jump description. The delay function

A deeper understanding of VSCPT can be obtained [Coh90, CBA91, Coh92] by applying to this problem the method of the delay function which has been developed for analysing photon correlations in resonance fluorescence [Rey83, CoD86], or the quantum jumps observable in the fluorescence light emitted by a single trapped ion [CoK85]. The initial motivation for this new investigation of subrecoil cooling was to overcome some difficulties of the numerical treatment of VSCPT by the GOBE at long interaction times θ. But the most important reason for briefly describing the results obtained by this approach is that they will lead us to the statistical model studied in this book, establishing the connection with Lévy statistics.

In most quantum optics problems, where an atom is driven by coherent laser fields and where the main source of dissipation comes from spontaneous emission, it is possible to describe the evolution of a single atom as a stochastic evolution consisting of a sequence of coherent evolution periods, involving absorption and stimulated emissions, separated by quantum jumps consisting of spontaneous emission processes which occur at random times. Such an evolution can also be interpreted in terms of a radiative cascade of the total system {atom + laser photons}, the so-called 'dressed atom' (see, for example, Chapter VI of [CDG88]). In this picture, each quantum jump is associated with a spontaneous transition of the dressed atom between two adjacent manifolds, by spontaneous emission of a fluorescence photon, and it puts the atom in a well defined state of the new manifold, correlated to the state of the spontaneously emitted photon. Between two quantum jumps, the dressed atom evolves within a given manifold. This coherent evolution is due to absorption and stimulated emission of laser photons by the atom, and it is described by a non-Hermitian effective Hamiltonian H_{eff}, which includes the effect of the damping due to spontaneous emission.

In this picture, it is natural to introduce the probability distribution $W(\tau)$ of the time intervals τ between two successive spontaneous emissions of photons by the same atom. Such a function, which is called the 'delay function' or the 'waiting time distribution', can be calculated by diagonalizing H_{eff} (see Appendix A.1). From $W(\tau)$ one can then deduce all the statistical properties of the random sequence of fluorescence photons. In particular, one can use $W(\tau)$ to make a Monte Carlo simulation of the random sequence of quantum jumps [CBA91, ZMW87]. Between two successive quantum jumps, the atom is described by its coherent evolution in a given manifold. Such a sequence of coherent evolutions separated

by random jumps constitutes a quantum jump simulation, and averaging over an ensemble of such simulations yields a quantum jump description of the process. Note that although the Monte Carlo drawings are based on usual (definite positive) probability laws such as $W(\tau)$, the quantum jump description is not classical, and contains all the quantum features of the evolution since these probability laws have been obtained from a quantum treatment of the problem. The quantum jump description thus provides a statistical description of all the interesting quantities.

We have performed quantum jump simulations of VSCPT, choosing an atomic transition and a one-dimensional laser configuration (corresponding to a real experiment [AAK88]), simple enough for allowing the delay function $W(\tau)$ to be calculated by diagonalizing a 3×3 matrix (only three internal states are involved) [CBA91, BBE94, Bar95]. In such a simple case, which has also been studied by using optical Bloch equations [AAK89], the coherent evolution periods take place within a three-dimensional atomic subspace $\mathcal{F}(p)$ (family of three atomic quantum states which are coupled by absorption and stimulated emission) labelled by using an index p which has the meaning of a momentum (the momentum of the system {atom + laser photons}). We call $W_p(\tau)$ the delay function obtained by diagonalizing H_{eff} within the subspace $\mathcal{F}(p)$. In a spontaneous emission process, the system changes randomly from a family $\mathcal{F}(p)$ to another one $\mathcal{F}(p')$ with $|p - p'| \leq \hbar k$, and we have to calculate a new delay function $W_{p'}(\tau)$. The time of the quantum jump which makes the atom leave $\mathcal{F}(p)$, and the corresponding change of atomic state, are obtained by a Monte Carlo procedure based on the delay function, and on the probability law of the change of atomic state, derived from a quantum analysis of the situation. Finally, the result of each simulation can be simply presented by giving the sequence of the constant values of p between two successive quantum jumps, and the times of the quantum jumps (see Fig. 2.1).

2.3.4 Simulation of the atomic momentum stochastic evolution

We have represented in Fig. 2.1, which is taken from [BBE94], an example of the stochastic evolution of the momentum p of an atom undergoing VSCPT, given by the quantum jump simulation described above.

At certain random times, the atom emits a photon and p changes abruptly. Between two successive spontaneous emissions, p remains constant. This simulation allows us to make the connection between the quantum description of VSCPT and the key ingredients of the cooling processes studied in this work. It clearly appears in Fig. 2.1 that the smaller p, the longer the delay τ between two successive spontaneous emissions: this is the principle of the new cooling schemes involving inhomogeneous random walks.

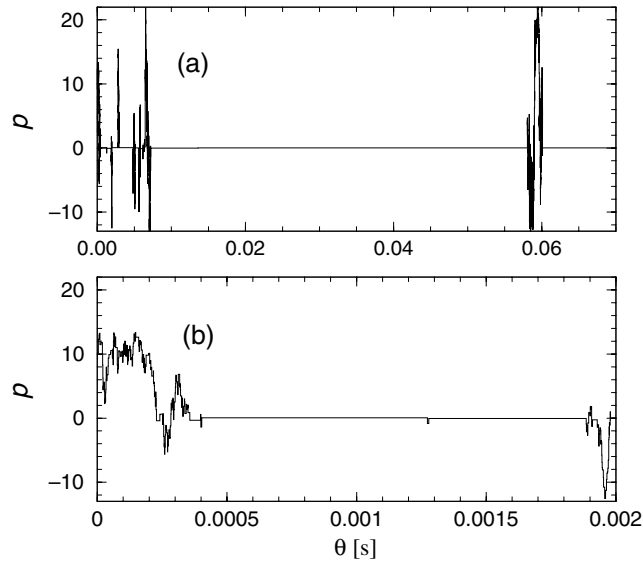

Fig. 2.1. (a) Example of a momentum random walk resulting from a quantum jump simulation of subrecoil cooling of metastable helium atoms. The unit of atomic momentum p is the momentum $\hbar k$ of the photons. The zoom (b) of the beginning of the time evolution is statistically analogous to the evolution at large scale. A striking point, typical of Lévy flights, is the fact that, for all time scales, the few longest time intervals dominate the evolution. Parameters: $\Omega_1 = 0.3\Gamma$ and $\delta = 0$ (see Appendix A for details).

There is another striking feature in Fig. 2.1, which was the starting point of the presented approach: the random sequence of time intervals is clearly dominated by a single term, the longest one, which is of the order of the total observation time. This feature gives an anomalous character to the random walk along the time axis. As we will see below, this statistically anomalous behaviour is the heart of the efficiency of subrecoil cooling and requires special statistical methods for its description.

2.3.5 Generalization. Stochastic wave functions and random walks in Hilbert space

More general one-dimensional configurations, where the delay function cannot be easily calculated, have also been investigated [MDT94] by using the so-called 'Monte Carlo Wave Function' (MCWF) method [DCM92, MCD93, MoC96], which in fact consists of a numerical calculation of the delay function, step by step. One can still consider families $\mathcal{F}(p)$ within which coherent evolution takes

place between two successive quantum jumps. For instance, in general laser configurations used in one-dimensional VSCPT, the laser electric field, which results from the superposition of two counterpropagating laser waves, varies periodically in space. The corresponding periodic optical potential in which the atom moves gives rise to a band structure of Bloch states [CaD91]. One can show [Coh96] that a given family $\mathcal{F}(p)$ is nothing but the set of Bloch states having the same quasi-momentum p. This gives a physical meaning to the index p labelling the families $\mathcal{F}(p)$. But these families are now atomic subspaces of infinite dimensions, and it is no longer possible to calculate the delay function by a simple diagonalization of a low dimension matrix. However, the resulting evolution can still be simply represented by a sequence of quantum jumps separated by coherent evolutions in well defined families labelled by a constant number p. Note again that there is no classical approximation in the above procedure, which retains all the quantum features of the problem.

The delay function approach and the MCWF method are both stochastic approaches belonging to a more general theoretical framework which is now being developed for analysing dissipative quantum optics problems and which is based on the idea of *stochastic wave functions*. There are a number of slightly different schemes [Car93, DZR92] which all share two basic ideas that distinguish stochastic wave functions from the traditional approach of optical Bloch equations. First, the atoms are no longer described by a density matrix but rather by an ensemble of wave functions. Second, the time evolution of these wave functions is not a fully continuous deterministic process as is the case for Bloch equations, but rather a sequence of coherent evolutions (continuous and deterministic) interrupted at random times by random instantaneous quantum jumps corresponding to spontaneous emissions.

Such a general stochastic wave function approach, which can be shown to be mathematically equivalent to Bloch equations, presents several interesting new features.

First, it provides a possible description of the quantum dissipative evolution of individual particles. Thus, by following the evolution of a single wave function, one gets a physical intuition of the quantum behaviour of a single atom, that would be harder to derive from the density matrix used in GOBE[2].

Second, the stochastic wave function approach can be efficiently implemented numerically. Indeed, in a problem where a continuous variable (like the momentum) is discretized in N steps with $N \gg 1$, it is much easier to perform calculations on wave functions of size N than on the corresponding density matrix of size

[2] For instance, this has been very useful for understanding the so-called 'dark periods' appearing in the fluorescence emitted by a single trapped ion [CoD86], or important features of certain schemes of amplification without inversion [CZA93].

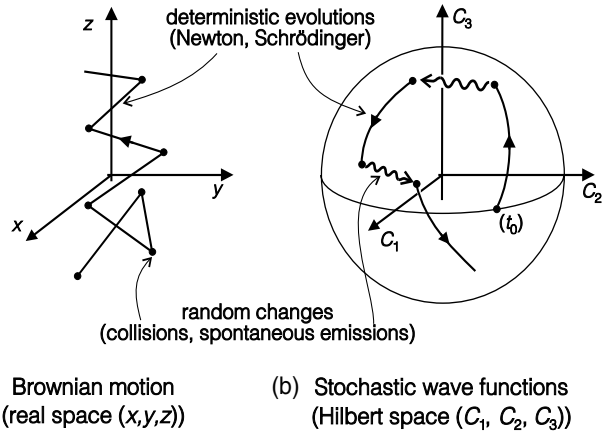

Fig. 2.2. Random walks: (a) of a classical particle in position space, under the influence of collisions with gas molecules interrupting free flights; (b) of a stochastic wave function in a Hilbert space (here three-dimensional), under the influence of spontaneous emissions interrupting Hamiltonian evolutions. Note that the wave function is normalized and therefore its motion is restricted to a sphere.

$N \times N$. This has proved to be crucial for the study of three-dimensional laser cooling [CaM95]. Another example is the analysis of the original one-dimensional VSCPT scheme: the explicit use of the delay function enables one to carry out calculations up to time scales 10 000 times larger than previous works with GOBE. Furthermore, the Monte Carlo algorithm exhibits multiscale properties that are especially well adapted for non-ergodic processes [Bar95]. Indeed the delay function allows one to choose the next spontaneously emitted photon in a single calculation step, even if the corresponding delay is very long. Since non-ergodic cooling is characterized by extremely long delays, this is a crucial advantage. In the case of more complex schemes, in which a step-by-step time integration replaces the delay function, the stochastic wave functions are still more tractable than the GOBE.

Third, stochastic wave functions provide a new description of quantum dissipative processes, which is very similar to a classical random walk (see Fig. 2.2). Indeed, between two spontaneous emissions, the atomic wave functions obey a coherent deterministic evolution, just as a Brownian particle obeys a free flight motion between two collisions. Then, the spontaneous emission is considered as instantaneous and random, occurring at random times and in random directions, just as collisions experienced by a Brownian particle. We are thus led to the important idea that a stochastic wave function performs a random walk in Hilbert space [BrP96]. Of course, the random walk of the wave function is more complicated than the random walk of a classical Brownian particle which is performed

in a classical position and momentum space. However, both random walks share an essential feature: they are sequences of deterministic evolutions separated by instantaneous random jumps. Therefore, the classical random walk techniques that have been developed with a high degree of sophistication should have some relevance for stochastic wave function approaches.

2.4 From quantum optics to classical random walks

A complete description of the random walk performed by the stochastic wave function in Hilbert space would be rather difficult and outside the scope of this work. We replace in the following such a full quantum analysis by a simpler description in terms of an inhomogeneous random walk performed by the momentum **p** of a fictitious classical particle. We keep in this classical description the important ingredients provided by the stochastic wave function approach, in order to be able to understand the basic features of the cooling process.

2.4.1 Fictitious classical particle associated with the quantum random walk

Let us come back to the quantum jump simulation of subrecoil cooling presented in Fig. 2.1. Such a random evolution looks like the random walk of a fictitious classical particle whose momentum p would change in a random way at certain times. It is clear that the evolution of such a classical particle cannot fully represent the quantum evolution of an ultracold atom in a subrecoil cooling experiment. Between two successive jumps, the state of the fictitious classical particle is fully characterized by a single number p, whereas the state of the ultracold atom is described by a wave function. Nevertheless, one must not forget that the simulation of Fig. 2.1 is derived from a rigorous quantum procedure. There are two ingredients of this quantum evolution which can be extracted and incorporated in the classical random walk, with the expectation that they could lead to a correct description of the cooling process.

First, the distribution of the delays between two successive spontaneous emissions is exactly calculated from an effective Hamiltonian. One can thus, at least in principle, exactly describe the distribution of the time intervals between two successive jumps performed by the fictitious classical particle. In practice, a further simplification will be introduced, as explained in the next section, which consists of taking simpler mathematical forms for the jump rate.

Second, in the simulation of Fig. 2.1, p is a constant of the motion of the quantum system between two successive jumps (the quasi-momentum characterizing the subspace of the Hilbert space where the coherent evolution takes place, i.e. the total momentum of the {atom+laser photons} system). If we consider this constant

of the motion as a physical quantity characterizing the fictitious classical particle, it will remain constant, as it should, between two 'collisions' experienced by such a particle.

Using the delay function and the constants of the motion of the quantum problem, we can thus introduce a classical random walk which reproduces a certain number of important features of the quantum random walk.

2.4.2 Simplified jump rate

The most important feature of the quantum jump simulation of Fig. 2.1 is the appearance of very long waiting times τ between two successive jumps when $\mathbf{p} \to \mathbf{0}$. We will thus take for the inhomogeneous random walk of the corresponding classical particle an average jump rate $R(\mathbf{p})$, which exactly vanishes for $\mathbf{p} = \mathbf{0}$ and which has a behaviour around this point characterized by the exponent α of the lowest-order term in the expansion of $R(\mathbf{p})$ in powers of $\mathbf{p}$. These features are extracted from the properties of the delay function $W_{\mathbf{p}}(\tau)$ which has a long tail in τ when $\mathbf{p} \to \mathbf{0}$, the corresponding decay rate decreasing as p^{α} when $p = \|\mathbf{p}\| \to 0$.

Strictly speaking, $W_{\mathbf{p}}(\tau)$ is the modulus of a sum of complex exponentials of τ. Our simplification consists of keeping only the exponential with the longest time constant, which tends to ∞ when $p \to 0$. Thus $W_{\mathbf{p}}(\tau)$ can be simply described by a jump rate $R(\mathbf{p})$:

$$W_{\mathbf{p}}(\tau) \simeq R(\mathbf{p}) \mathrm{e}^{-R(\mathbf{p})\tau}. \tag{2.6}$$

This approximation can be justified by the following argument. The important point in the cooling schemes described here is the fact that the quantum system arriving after a jump in a family $\mathcal{F}(\mathbf{p})$ with $\mathbf{p}$ close to 0 has a non-zero probability to remain there for a very long time (which tends to ∞ when $p \to 0$). It could eventually make a jump after a very short time, because $W_{\mathbf{p}}(\tau)$ also contains rapidly decaying exponential components but, after a certain time, the system will come back in the neighbourhood of $\mathbf{p} = \mathbf{0}$, one or several times, until it stays there for a very long time. Such very long sojourn times are the origin of the 'anomalous' character of the random simulation of Fig. 2.1. By keeping only the longest time constant in $W_{\mathbf{p}}(\tau)$, one can hope to keep this essential ingredient which will allow us to derive correctly the asymptotic properties of the cooled atoms. Including the other shorter time constants would change only some prefactors[3], but would not modify the asymptotic θ-dependence of the various physical quantities.

Similarly, we will characterize the classical random walk far from $\mathbf{p} = \mathbf{0}$ by the behaviour of $R(\mathbf{p})$ for larger values of $\mathbf{p}$. We will introduce in Chapter 3 simple

[3] See Appendix A, in particular eq. (A.35), where we derive these prefactors for one-dimensional VSCPT.

models corresponding to different possible physical situations. Here also, we will find that there are situations leading to anomalous random walks.

2.4.3 Discussion

The basic idea of this simplified model is that the efficiency of subrecoil cooling is linked to the slowing down of the random walk around $\mathbf{p} = \mathbf{0}$, but that details of the exact characteristics of the random walk are unimportant. We are in fact following the usual approach in statistical physics, where general and powerful results can be found independently of microscopic details, provided that some essential features are taken into account. It seems difficult to demonstrate rigorously the validity of this approach, but of course we will compare its results to the results of the quantum microscopic calculations in the cases where such results are available, either from a GOBE treatment, or from a quantum jump treatment.

A benefit of our statistical approach is to yield quantitative predictions, even in cases where the quantum microscopic treatment is unable to make such predictions. It will thus enable us to address a few important questions such as the asymptotic behaviour at very long interaction times ($\theta \to \infty$) or the efficiency of subrecoil cooling in a configuration of dimension D larger than one.